Gauss' Theorem

- Is Area is a Vector OR ~~Scalar~~ ?

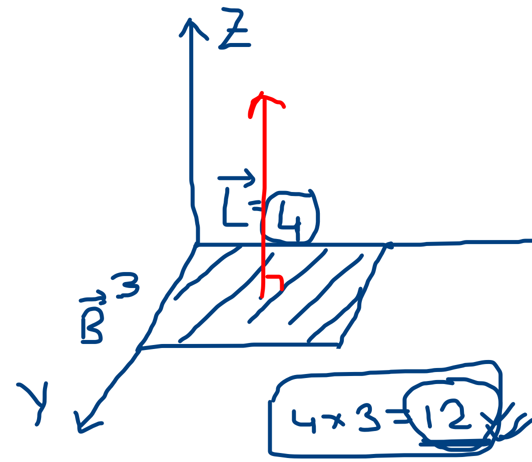
$$\vec{L} = 4\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\vec{B} = 0\hat{i} + 3\hat{j} + 0\hat{k}$$

$$\vec{L}\cdot\vec{B} = 0 + 0 + 0 = 0 \quad \times$$

$$\vec{L}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 0 & 0 \\ 0 & 3 & 0 \end{vmatrix} = (0-0)\hat{i} + (0-0)\hat{j} + (12-0)\hat{k}$$

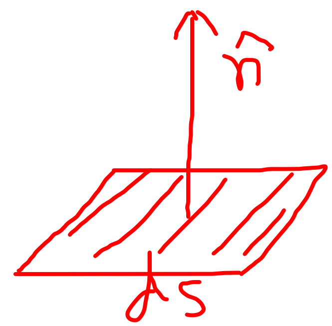
$$= 0\hat{i} + 0\hat{j} + 12\hat{k}$$



Area vector: An area element dS can be represented as $\overrightarrow{dS}$.

It is defined as a vector whose magnitude is dS and its direction is unit vector $\hat{n}$ perpendicular to surface dS.

$$\boxed{\overrightarrow{dS} = dS\,\hat{n}}$$



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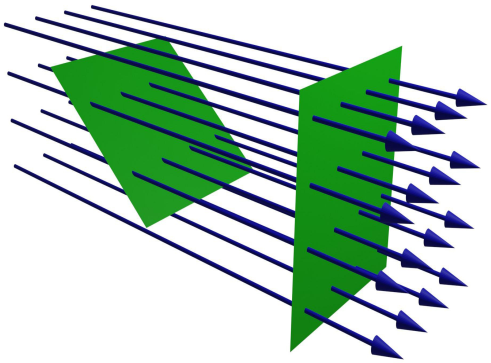
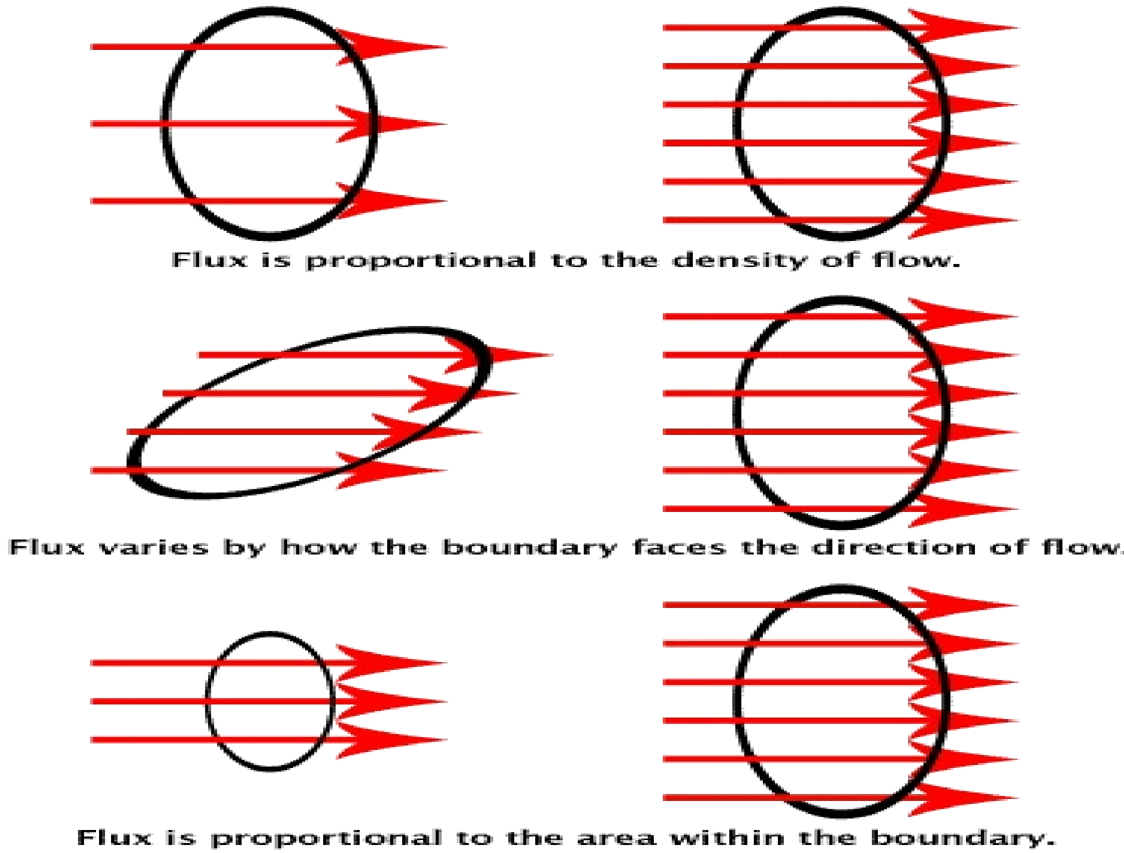
Electric Flux:





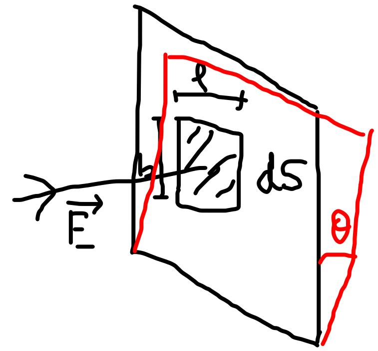
Electric flux is defined total number of electric field lines that normally pass through that surface. It is represented by Φ_E.

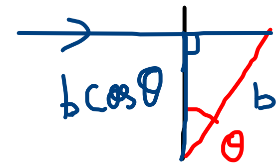
$ds = l \times b$

$ds' = \boxed{l \times b} \cos\theta$

$= ds \cos\theta$

$E \; ds$





So electric flux through small area dS is given by,

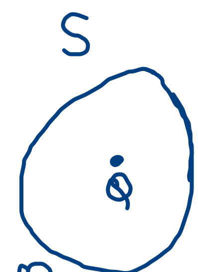
$$d\Phi_E = \vec{E} dS \cos\theta = \vec{E} . d\vec{S}$$

For a whole surface S due to an electric field E, total electric flux is given by,

$$\Phi_E = \oint_S \vec{E} . d\vec{S} = \oint_S \vec{E} \, d\vec{S} \cos\theta$$

SI unit of Electric Flux is **Nm^2C^{-1}**

Dimensional formula of $\mathbf{\Phi_E}$ is $[M^1L^1T^{-2}][L^2][A^1T^1]^{-1} = \left[\boldsymbol{M^1L^3T^{-3}A^{-1}}\right]$



$\frac{1}{\epsilon_0} Q$

Statement of Gauss' theorem:

It states that total electric flux over the closed surface S in vacuum is $\frac{1}{\epsilon_0}$ times the total charge(Q) contained inside S.

$$\text{i.e. } \Phi_E = \oint_S \vec{E} . d\vec{S} = \frac{1}{\epsilon_0} Q$$

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Proof of Gauss' theorem

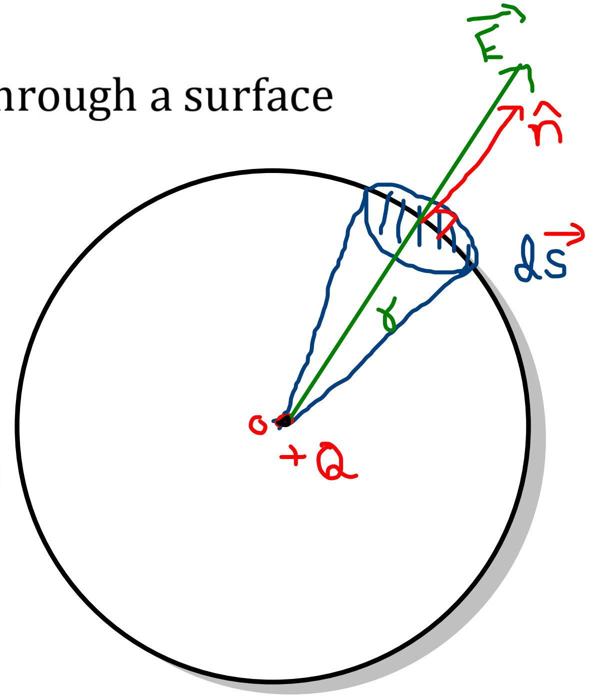
Let us consider a spherical surface of radius r, electric flux through a surface element dS is given by

$$d\Phi_E = \vec{E}.d\vec{S} = \vec{E}\, d\vec{S}\, Cos\theta$$

$$d\Phi_E = EdS \qquad (\because \theta = 0)$$

Or,

$$d\Phi_E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}dS$$



Therefore, total electric flux through the spherical surface is

$$\Phi_E = \oint_S d\Phi_E = \oint_S \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}dS$$

$$\Phi_E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\oint_S dS$$

$$\Phi_E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,(4\pi r^2)$$

$$\boxed{\Phi_E = \frac{Q}{\epsilon_0}}$$

Gaussian Surface and its properties:

A gaussian surface is an arbitrary (imaginary) closed surface in 3-dimensional ~~surface~~ Space through which flux of vector field is calculated.

Examples: Sphere, cylinder, cube……etc.

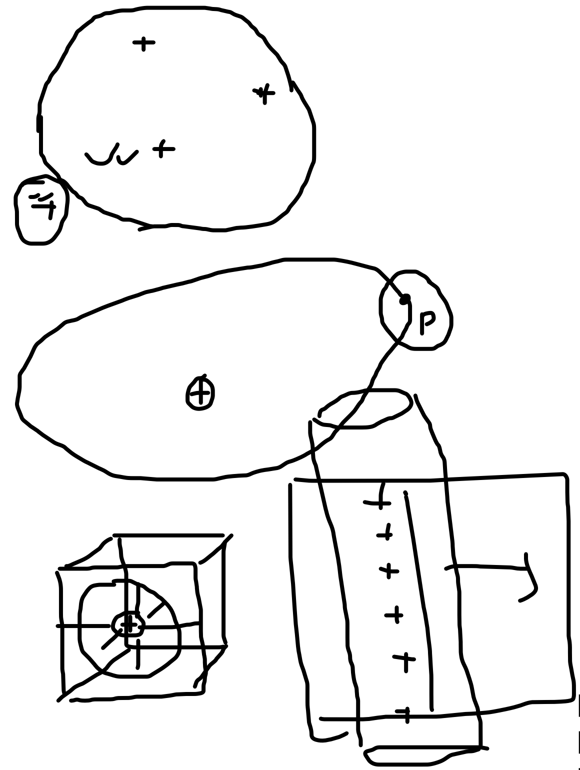
not valid: disc surface, square, circle……

Properties:

1. It should be closed surface so that the clear distinction can be made between points that are inside or outside the surface.

2. This surface must pass through the point where electric field is to be calculated.

3. Surface must have shape according to symmetry of the source. So that field is normal to the surface at each point and constant in magnitude.



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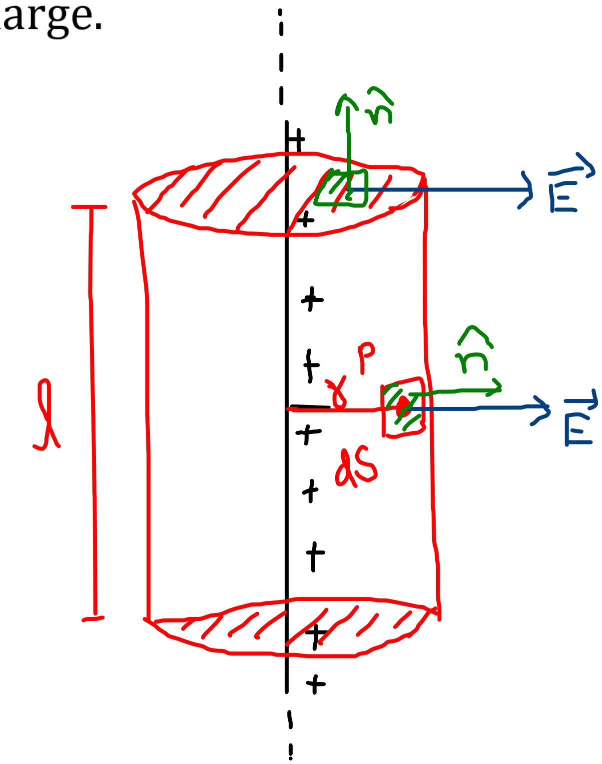
4. For a system of charge the Gaussian surface should not pass through any discrete charge. It is because electric field at the location of any charge is not well defined. However, the Gaussian surface can pass through a continuous charge distribution.

Gauss' theorem is useful in computing the electric field due to system of charges or symmetrical continuous distribution of charge.

<u>Application of Gauss' theorem:</u>

i) <u>Electric field due to an infinite line of charge</u>

Let λ be the linear charge density and P be the point at which the electric intensity has to be calculated. Let us draw a coaxial Gaussian cylindrical surface of length l and radius r, through point P.



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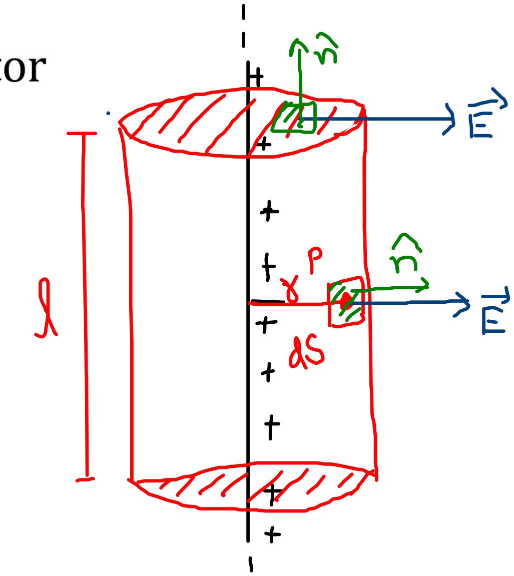
Thus, for any area element dS taken on surface both the electric field vector $\vec{E}$ and area vector $d\vec{S}$ are along the same direction i.e. $\theta = 0$ therefore,

$$\vec{E}.d\vec{S} = \vec{E}\,d\vec{S}\,Cos\theta = E\,dS$$



Hence electric flux through curved surface of cylinder is,

$$\Phi_{E_c} = \oint_S \vec{E}\,.\,d\vec{S} = E\oint_S dS = E(2\pi r l)$$

now, the flux through the plane end of the surface is Zero,

i.e. $\Phi_{E_p} = 0$ $\quad$ ($\because$ $\vec{E}$ and $d\vec{S}$ is perpendicular at plane end)

Total flux through the gaussian surface is:

$$\Phi_E = \Phi_{E_C} + \Phi_{E_p} = E(2\pi\, r\, l)\text{--------------------(i)}$$

using Gauss' theorem,

$$\Phi_E = \frac{Q}{\epsilon_0} \qquad \text{----------------------------(ii)}$$

where, Q is total charge, $Q = \lambda l$

from eq (i) and (ii)

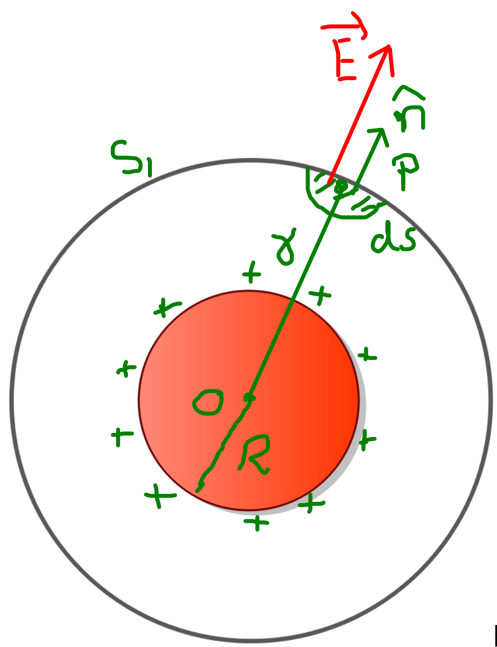
$$E(2\pi\, r\, l) = \frac{Q}{\epsilon_0}$$

$$E = \frac{\lambda \cancel{l}}{2\pi\epsilon_0\, r \cancel{l}}$$

$$E = \frac{\lambda}{2\pi\epsilon_0\, r}$$

In vectors, $\boxed{\vec{E} = \frac{\lambda}{2\pi\epsilon_0\, r}\hat{r}}$

ii) <u>Electric field due to a uniformly charged spherical shell</u>



Let us consider a spherical shell of radius R with center at O. Let the charge +q is distributed uniformly over the surface of the shells, and have surface charge density σ.
To calculate electric field at point P which is at **r** distance from centre O, we need to draw a spherical Gaussian surface S_1.

a) <u>electric field outside the spherical shell</u>

now, According to Gauss' theorem,

$$\Phi_E = \frac{q}{\epsilon_0}$$

$$\Rightarrow \quad \oint_S \vec{E} \, . \, d\vec{S} = \frac{q}{\epsilon_0}$$

$$\Rightarrow \quad \oint_S \underline{E} \; \underline{dS} = \frac{q}{\epsilon_0} \qquad (\because \; \vec{E} \text{ and } d\vec{S} \text{ are in same direction})$$

$$\Rightarrow \quad E \oint_S dS = \frac{q}{\epsilon_0}$$

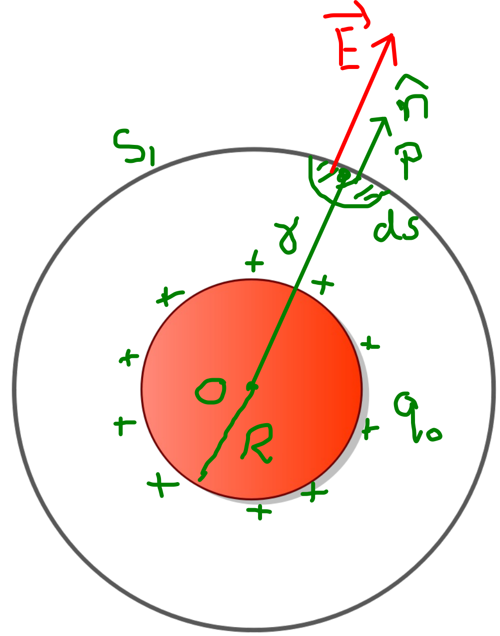
$$\Rightarrow \quad E\,(4\pi r^2) = \frac{q}{\epsilon_0}$$

then, $$\boxed{E = \frac{q}{(4\pi\epsilon_0 r^2)}} \text{ -------------------------(i)}$$

and, $\sigma = \dfrac{q}{4\pi R^2}$

Or, $q = \sigma \, 4\pi R^2$

$$\therefore \quad \boxed{E = \frac{\sigma R^2}{\epsilon_0 r^2}} \text{ ---------------------------------(ii)}$$



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b) electric field on the surface of the shell (r=R)

putting r= R in Eq (ii), we can write

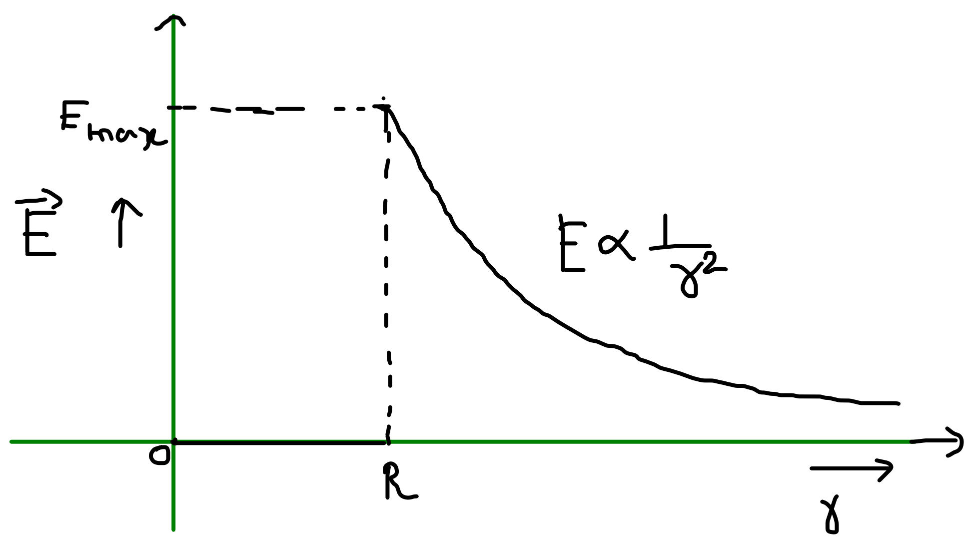
$$E = \frac{\sigma \cancel{R^2}}{\epsilon_0 \cancel{R^2}}$$

$$\boxed{\therefore E = \frac{\sigma}{\epsilon_0}}$$

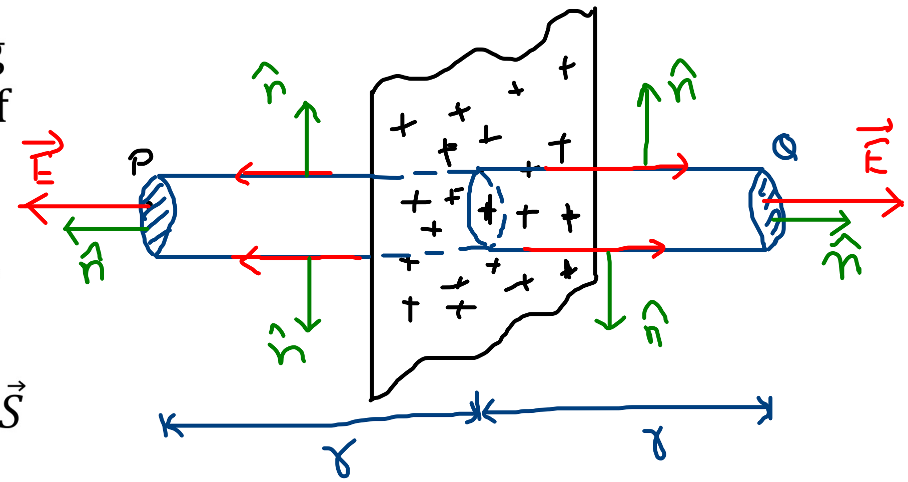
c) If point P is inside the spherical shell

Gaussian surface for inside the spherical shell contains(encloses) no charge, i.e. q=0

Therefore, E=0.



iii) Electric field due to infinite plane sheet of charge



Let us consider a thin infinite plane sheet having charge +q. Let σ be the surface charge density of the sheet.

To calculate electric field intensity at P, distant r from sheet.

Let us imagine a cylinder of cross-section area $d\vec{S}$ around P of length 2r.

At P and Q, $\vec{E}$ and $\hat{n}$ are parallel to each other, i.e. $\theta = 0^o$

so, electric flux at P,

$$\Phi_{E_P} = \vec{E}.d\vec{S} = \vec{E}\ d\vec{S}\ Cos\ 0^o = E\ dS$$

Similarly, at Q

$$\Phi_{E_Q} = E\ dS$$

and, on curved surface area of gaussian cylinder $\vec{E}$ is perpendicular to $d\vec{S}$ so electric flux at curved surface area is,

$$\Phi_{E_C} = E\ dS\ Cos\ 90^o = 0$$

hence total electric flux over entire surface of cylinder is

$$\Phi_E = \Phi_{E_P} + \Phi_{E_Q} + \Phi_{E_C}$$

$$\Phi_E = 2E\ dS \text{ ------------------------ } \quad \text{(i)}$$

According to gauss theorem,

$$\Phi_E = \frac{q}{\epsilon_0}$$

$$\sigma = \frac{q}{ds}$$

$$\text{Or, } \Phi_E = \frac{\sigma\ dS}{\epsilon_0} \text{ ---------------------- } \quad \text{(ii)}$$

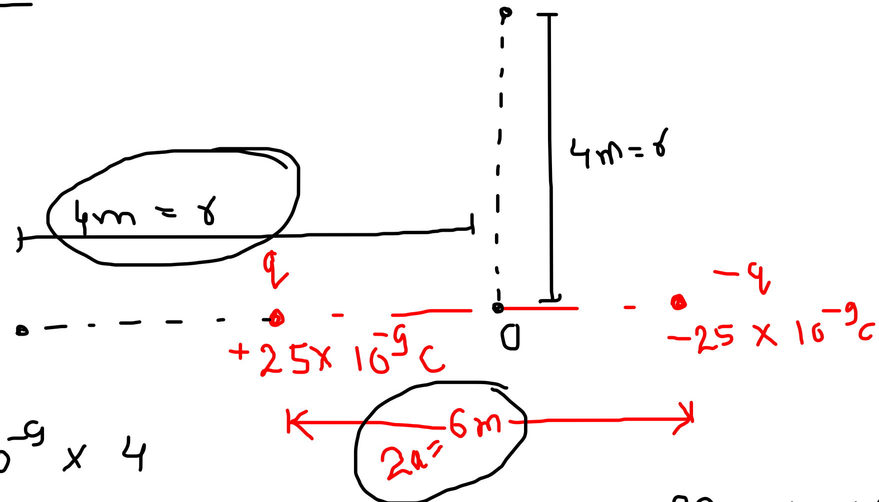
$$q = \underline{\sigma\ ds}$$

from (i) and (ii)

$$2E\ \cancel{dS} = \frac{\sigma\ \cancel{dS}}{\epsilon_0} \quad \Rightarrow \boxed{E = \frac{\sigma}{2\epsilon_0}}$$

Electric field is independent of **r.**

Que 15: Two charges of + 25 X 10^{-9} C and -25 X 10^{-9} C are placed 6 m apart. Find the electric field at a point 4m from the centre of the electric dipole (i) on axial line (ii) on equatorial line.



i) $E_a = \frac{1}{4\pi\epsilon_0} \frac{2pr}{(r^2 + a^2)^2}$

$|\vec{p}| = 25 \times 10^{-9} \times 6$

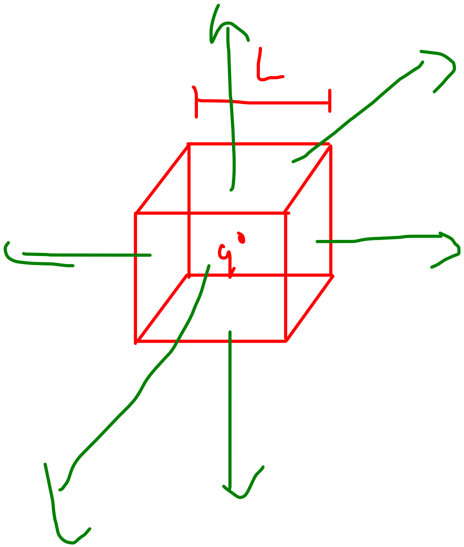
$\vec{p} = 150 \times 10^{-9}$ C m

$E_a = 9 \times 10^{9} \times \frac{2 \times 150 \times 10^{-9} \times 4}{(4^2 + 3^2)^2} = \; ? \; = 220.4$ N/C

(ii) $E_E = \frac{1}{4\pi\epsilon_0} \frac{p}{(r^2 + a^2)^{3/2}} = \; ? \; = 10.8$ N/C

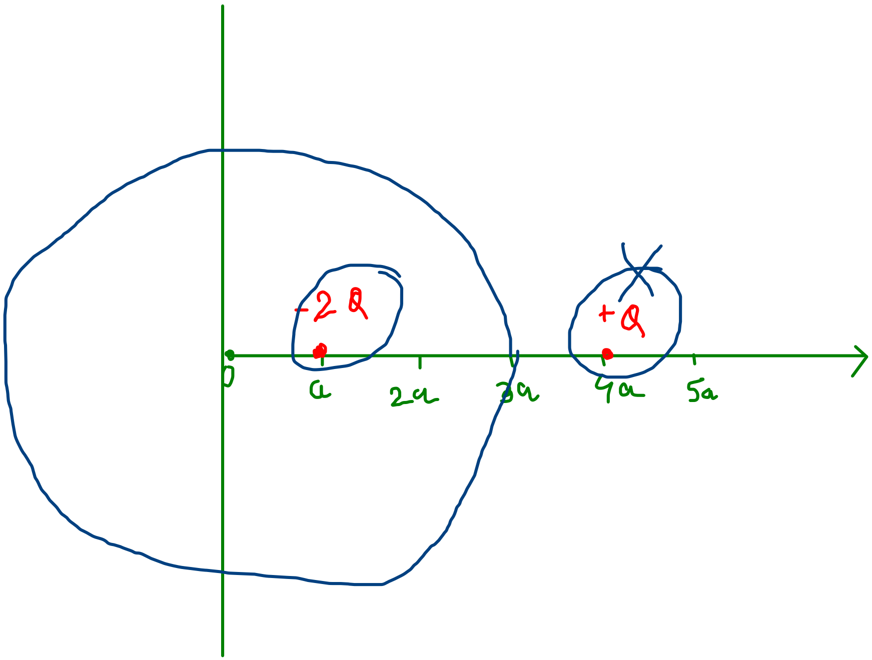
Que 16: A charge q is placed at the centre of a cube of side L. What is the electric flux passing through each face of the cube?

$$\boxed{\phi_E = \frac{1}{6}\frac{q}{\epsilon_0}}$$



Que 17: Two charges of magnitudes -2Q and +Q located at point (a,0) and (4a,0), respectively. What is the electric flux due to these charges through a sphere of radius 3a with its center at origin?

$$\boxed{\phi_E = \frac{-2Q}{\epsilon_0}}$$



Que 18: Intensity of electric field at a perpendicular distance 0.5m from an infinitely long line charge having linear charge density (λ) is 3.6×10^3 V m^{-1}. Find the value of λ.

$$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{r}$$

$$\Rightarrow \quad 3.6 \times 10^3 = 9 \times 10^9 \frac{2\lambda}{0.5}$$

$$= ?$$

$$\boxed{\lambda = 10^{-7}\ C/m}$$

Que 19: An infinite line charge produces a field of 9 X 10^4 N C^{-1} at a distance of 2 cm. Calculate the line charge density. (λ)

-- Same as Que 18----
do it yourself

Que 20 : A point charge +10μ C is at a distance 5 cm directly above the centre of a square of side 10 cm. What is the magnitude of the electric flux through the square?

$$\phi = \frac{1}{6}\frac{q}{\epsilon_0} = \frac{1}{6}\frac{10\times10^{-6}}{8.85\times10^{-12}}$$

$$= \;?$$